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Invariant Finsler metrics on homogeneous manifolds*

Shaoqiang Deng and Zixin Hou

School of Mathematics and LPMC, Nankai University, Tianjin, People's Republic of China

E-mail: dengsq@nankai.edu.cn and houzx@nankai.edu.cn

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Abstract
In this paper, we study invariant Finsler metrics on homogeneous manifolds. We first give an algebraic description of these metrics and obtain a necessary and sufficient condition for a homogeneous manifold to have invariant Finsler metrics. As a special case, we study bi-invariant Finsler metrics on Lie groups and obtain a necessary and sufficient condition for a Lie group to have bi-invariant Finsler metrics. Finally, we provide some conditions for a homogeneous manifold to admit invariant non-Riemannian Finsler metrics and present some interesting examples.

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Introduction

The study of Finsler spaces has important significance in physics. In [1], the authors single out four aspects of the contexts in which the integral $\int_a^b F(x, y)$ of a Finsler metric $F(x, y)$ arises, three of them being related to physics. Take optics for instance. In an anisotropic medium, the speed of light depends on its direction of travel. At each location x, visualize y as an arrow that emanates from x. Measure the time light takes to travel from x to the tip of y, and call the result $F(x, y)$. Then $\int_a^b F(x, y)$ represents the total time light takes to traverse a given path in this medium (cf also [2]). Meanwhile, some authors also pointed out that Einstein's general relativity theory can be described at a more accurate level using Finsler geometry instead of Riemannian geometry (cf [3]).

However, it is generically very difficult to construct an explicit non-Riemannian Finsler space, except for some trivial examples, such as Minkowski or locally Minkowski spaces. Therefore, it is very important to find an effective way to construct explicit examples of Finsler spaces. This paper will present an algebraic method to create homogeneous Finsler spaces.

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Let G be a Lie group, H be a closed subgroup of G. The coset space G/H has a unique smooth (analytic) structure such that G is a Lie transformation group of G/H. It is called reductive if there exists a subspace $\mathfrak{m}$ of the Lie algebra $\mathfrak{g} = \text{Lie } G$ such that

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} \qquad \text{(direct sum of subspaces)},$$

where $\mathfrak{h} = \text{Lie } H$ and $\text{Ad}(h)\mathfrak{m} \subset \mathfrak{m}, \forall h \in H$.

The study of invariant structures on reductive coset spaces is an important problem in geometry. Nomizu's research on the properties of invariant Riemannian metrics on G/H obtained many interesting and significant results. He computed the connections of these metrics and obtained the formula for geodesics and curvatures. His study created many significant examples of Riemannian manifolds which have many special properties (cf [4]).

Therefore it is important to study invariant Finsler metrics on homogeneous manifolds, especially reductive homogeneous manifolds. In this paper, we first give an algebraic description of these structures. We first give the definition of Minkowski Lie pairs (definition 1.1) and show that an invariant Finsler metric on G/H will induce a Minkowski Lie pair and the converse is also true if H is connected. As a special case, we study bi-invariant Finsler metrics on Lie groups and introduce the notion of Minkowski Lie algebras (definition 2.1) to describe such structures. In section 3, we give a formula for the geodesics, connections and flag curvatures in some special cases. Finally, we obtain some condition when there exist invariant non-Riemannian Finsler metrics on G/H and present some examples. The main result of this paper is the promotion of our previous paper on invariant Randers' metrics on homogeneous Riemannian manifolds (cf [5]).

We must point out that the notions of Minkowski Lie pairs and Minkowski Lie algebras seem to be the best combination of the related notions of functional analysis, Lie theory and differential geometry. Hopefully, these new notions will play important roles in these related fields, as well as in the applications of these subjects to physics.

1. Algebraic description

In this section, we give an algebraic description of the invariant Finsler metrics on homogeneous manifolds (not necessarily reductive).

Proposition 1.1. *Let G be a Lie group, H a closed subgroup of G, Lie $G = \mathfrak{g}$, Lie $H = \mathfrak{h}$. Then there is a one-to-one correspondence between the G-invariant Finsler metric on G/H and the Minkowski norm on the quotient space $\mathfrak{g}/\mathfrak{h}$ satisfying*

$$F(\text{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)(x)) = F(x), \qquad \forall h \in H, \quad x \in \mathfrak{g}/\mathfrak{h},$$

where $\text{Ad}_{\mathfrak{g}/\mathfrak{h}}$ is the representation of H on $\mathfrak{g}/\mathfrak{h}$ induced by the adjoint representation of H on $\mathfrak{g}$.

Proof. The proof is similar to the Riemannian case, so we omit it (cf [6]). □

Corollary 1.2. *Let G/H be a reductive homogeneous manifold with the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Then there is a one-to-one correspondence between the G-invariant Finsler metric on G/H and the Minkowski norm on $\mathfrak{m}$ satisfying*

$$F(\text{Ad}(h)x) = F(x), \qquad \forall h \in H, \quad x \in \mathfrak{m}.$$

To state the next result, we first give a definition.

Definition 1.1. *Let $\mathfrak{g}$ be a real Lie algebra, $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. If F is a Minkowski norm on the quotient space $\mathfrak{g}/\mathfrak{h}$ such that*

$$g_y(\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}(x)u, v) + g_y(u, \mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}(x)v) + 2C_y(\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}(x)(y), u, v) = 0,$$

where $y, u, v \in \mathfrak{g}/\mathfrak{h}$, $y \neq 0$, $x \in \mathfrak{h}$, g_y is the (positive definite) inner product defined by F at y, and C_y is the Cartan tensor of F (see [1] for the definitions). Then $\{\mathfrak{g}, \mathfrak{h}, F\}$ (or simply $\{\mathfrak{g}, \mathfrak{h}\}$) is called a Minkowski Lie pair. *In particular, if F is an Euclidean norm (this is the case if and only if for any y, $C_y = 0$, cf [1]), then $\{\mathfrak{g}, \mathfrak{h}\}$ is called an* Euclidean Lie pair.

Theorem 1.3. *Let G be a Lie group, H a closed subgroup of G, Lie $G = \mathfrak{g}$, Lie $H = \mathfrak{h}$. Suppose there exists an invariant Finsler metric on the homogeneous manifold G/H. Then there exists a Minkowski norm F on $\mathfrak{g}/\mathfrak{h}$ such that $\{\mathfrak{g}, \mathfrak{h}, F\}$ is a Minkowski Lie pair. On the other hand, if H is connected and there exists a Minkowski norm on $\mathfrak{g}/\mathfrak{h}$ such that $\{\mathfrak{g}, \mathfrak{h}, F\}$ is a Minkowski Lie pair, then there exists an invariant Finsler metric on G/H.*

Proof. Let F be an invariant Finsler metric on G/H. Identifying $\mathfrak{g}/\mathfrak{h}$ with $T_o(G/H)$, where o is the origin of G/H, we get a Minkowski norm on $\mathfrak{g}/\mathfrak{h}$ (still denoted by F). Since F is G-invariant, we have

$$F(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)(u)) = F(u), \qquad \forall h \in H, \quad u \in \mathfrak{g}/\mathfrak{h}.$$

By the definition of g_y, we have

$$g_y(u, v) = g_{\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)(y)}(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)(u), \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)(v)), \qquad \forall y, u, v \in \mathfrak{g}/\mathfrak{h}, \quad y \neq 0, \quad h \in H.$$

For any $x \in \mathfrak{h}$, the one-parameter subgroup $\exp tx$ of H gives

$$g_y(u, v) = g_{\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\exp tx)(y)}(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\exp tx)(u), \qquad \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\exp tx)(v)), \quad \forall t \in \mathbb{R}.$$

Taking the derivative with respect to t, we obtain

$$g_y(\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}(x)u, v) + g_y(u, \mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}(x)v) + 2C_y(\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}(x)(y), u, v) = 0.$$

Therefore $\{\mathfrak{g}, \mathfrak{h}, F\}$ is a Minkowski Lie pair. On the other hand, suppose F is a Minkowski norm on $\mathfrak{g}/\mathfrak{h}$ such that $\{\mathfrak{g}, \mathfrak{h}, F\}$ is a Minkowski Lie pair. For any $x \in \mathfrak{h}$, $y, u, v \in \mathfrak{g}/\mathfrak{h}$, $y \neq 0$, consider the function

$$\psi(t) = g_{\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\exp tx)(y)}(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\exp tx)(u), \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\exp tx)(v)).$$

Then for any $t_0 \in \mathbb{R}$ we have

$$\begin{aligned}\psi'(t_0) = {} & g_{\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\exp t_0x)(y)}(\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}(t_0x)(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\exp t_0x)(u)), \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\exp t_0x)(v)) \\ & + g_{\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\exp t_0x)(y)}(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\exp t_0x)(u), \mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}(x)(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\exp t_0x)(v))) \\ & + 2C_{\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\exp t_0x)(y)}(\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}(x)(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\exp t_0x)(y)), u, v) = 0.\end{aligned}$$

Therefore the function ψ is a constant, hence

$$g_y(u, v) = g_{\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\exp tx)(y)}(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\exp tx)(u), \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(\exp tx)v), \qquad \forall t \in \mathbb{R}.$$

Since H is connected, it is generated by elements of the form $\exp tx$, $x \in \mathfrak{h}$, $t \in \mathbb{R}$. Therefore for any $h \in H$, we have

$$g_y(u, v) = g_{\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)(y)}(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)(u), \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)v).$$

To complete the proof, we need some computation. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be a basis of the linear space $\mathfrak{g}/\mathfrak{h}$. For $u \in \mathfrak{g}/\mathfrak{h} - \{0\}$, define $g_{ij}(u) = g_u(\alpha_i, \alpha_j)$. Then we have the formula (cf [1])

$$F^2(u) = \sum_{i,j=1}^{n} g_{ij}(u)u^i u^j,$$

where $u = \sum u^i \alpha_i$. Now for any $h \in H$, we have

$$F^2(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)(u)) = \sum_{i,j=1}^{n} g_{ij}(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)(u))\bar{u}^i\bar{u}^j,$$

where $\bar{u}^i (i = 1, 2, \ldots, n)$ are defined by $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)(u) = \sum \bar{u}^i \alpha_i$. Let $(m_{ij})_{n\times n}$ be the matrix of $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)$ under the basis $\alpha_1, \ldots, \alpha_n$, and $(m^{ij})_{n\times n}$ be the inverse of (m_{ij}). Then

$$\bar{u}^i = \sum_{k=1}^{n} m_{ik} u^k.$$

Now

$$\begin{aligned} g_{ij}(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)(u)) &= g_{\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)(u)}(\alpha_i, \alpha_j) \\ &= g_u((\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h))^{-1}\alpha_i, (\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h))^{-1}\alpha_j) \\ &= g_u\left(\sum_{k=1}^{n} m^{ik}\alpha_k, \sum_{l=1}^{n} m^{jl}\alpha_l\right). \end{aligned}$$

Therefore, taking into account the fact that g_u is bilinear, we have

$$\begin{aligned} F^2(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)(u)) &= \sum_{i,j=1}^{n} g_u\left(\sum_{k=1}^{n} m^{ik}\alpha_k, \sum_{l=1}^{n} m^{jl}\alpha_l\right)\left(\sum_{s=1}^{n} m_{is}u^s\right)\left(\sum_{t=1}^{n} m_{jt}u^t\right) \\ &= \sum_{i,j=1}^{n} g_u(\alpha_i, \alpha_j)u^i u^j. \end{aligned}$$

Thus

$$F^2(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)(u)) = F^2(u).$$

Since $F \geqslant 0$, we have

$$F(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)(u)) = F(u).$$

Therefore, by the correspondence of proposition 1.1, we see that there exists an invariant Finsler metric on G/H. □

To find a necessary and sufficient condition for G/H to have invariant Finsler metrics, we first make an observation.

Theorem 1.4. *Let G be a Lie group, H be a closed subgroup of G. Suppose there exists an invariant Finsler metric on G/H. Then there exists an invariant Riemannian metric on G/H.*

Proof. Let $o = H$ be the origin of G/H. Consider the tangent space $T_o(G/H)$ of G/H at o. Then F defines a Minkowski norm on $T_o(G/H)$ (still denoted by F). Define

$$I_o = \{x \in T_o(G/H) | F(x) = 1\}.$$

Then the linear isotropic group $\mathrm{Ad}(H) = \{\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h) | h \in H\}$ leaves I_o invariant. Let G_1 be the subgroup of the general linear group $GL(T_o(G/H))$ consisting of the elements which leave I_o invariant. Then G_1 is a compact Lie group (cf [6]) and $\mathrm{Ad}(H)$ is a subgroup of G_1. Therefore we can choose a G_1-invariant inner product $\langle\, ,\rangle$ on $T_o(G/H)$. Then $\langle\, ,\rangle$ is $\mathrm{Ad}\,(H)$-invariant. Therefore, using $\langle\, ,\rangle$, we can define a G-invariant Riemannian metric g on G/H (cf [6]). □

Using the well-known result on homogeneous Riemannian manifolds, we have

Corollary 1.5. *Let G be a Lie group, H be a closed subgroup of G, Lie $G = \mathfrak{g}$, Lie $H = \mathfrak{h}$. Suppose there exists a G-invariant Finsler metric on G/H and $\mathfrak{h}$ contains no non-zero ideal of $\mathfrak{g}$. Then*

- *The Killing form $B_{\mathfrak{h}}$ of $\mathfrak{h}$ is semi-negative definite.*
- *The restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$ is negative definite.*

Now we can give a necessary and sufficient condition for a homogeneous manifold to have invariant Finsler metrics.

Theorem 1.6. *Let G be a Lie group, H be a closed subgroup of G such that G acts effectively on G/H. Suppose the centralizer of H in G is non-discrete. Then there exists an invariant Finsler metric on G/H if and only if there exists a Minkowski norm F on the Lie algebra $\mathfrak{g}$ of G such that*

$$F(\mathrm{Ad}(h)x) = F(x), \qquad \forall h \in H, \quad x \in \mathfrak{g}.$$

Proof. The necessity is obvious, since if there exists an invariant Finsler metric on G/H, then by theorem 1.4, there exists an invariant Riemannian metric on G/H. Therefore there exists an inner product $\langle\, ,\, \rangle$ on $\mathfrak{g}$ such that (cf [6])

$$\langle \mathrm{Ad}(h)x, \mathrm{Ad}(h)y\rangle = \langle x, y\rangle, \qquad \forall h \in H, \quad x, y \in \mathfrak{g}.$$

Define $F(x) = \sqrt{\langle x, x\rangle}$. Then F satisfies our condition.

Now we prove the sufficiency. Suppose there exists a Minkowski norm on $\mathfrak{g}$ satisfying $F(\mathrm{Ad}(h)x) = F(x), h \in H, x \in \mathfrak{g}$. Then by theorem 1.3, for any $y \in \mathfrak{g}, y \neq 0$, we have

$$\begin{aligned} g_y(\mathrm{Ad}(h)(x), \mathrm{Ad}(h)(y)) &= \frac{1}{2}\frac{\partial^2}{\partial s \partial t} F^2(y + s\mathrm{Ad}(h)x + t\mathrm{Ad}(h)y)|_{s=t=0} \\ &= g_{\mathrm{Ad}(h^{-1})y}(x, y), \end{aligned}$$

where $h \in H, x, y \in \mathfrak{g}$. Since $C_H(G)$ is non-discrete, we can find a non-zero $y \in \mathfrak{g}$ such that $\mathrm{Ad}(h)y = y, \forall h \in H$. Then the inner product g_y satisfies

$$g_y(\mathrm{Ad}(h)x, \mathrm{Ad}(h)y) = g_y(x, y).$$

Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to g_y. Then we have

$$\mathrm{Ad}(h)\mathfrak{m} \subset \mathfrak{m},$$

and

$$F(\mathrm{Ad}(h)x) = F(x), \qquad \forall h \in H, \quad x \in \mathfrak{m}.$$

By corollary 1.2, there exist invariant Finsler metrics on G/H. □

Remark. The condition that the centralizer of H in G is non-discrete is indispensable and is easy to satisfy.

2. Bi-invariant Finsler metrics on Lie groups

In this section, we consider invariant Finsler metrics on Lie groups. Let G be a Lie group. Then we can write $G = G/H$ with $H = \{e\}$ and the action is the left translation of G. Therefore, by proposition 1.1, for every Minkowski norm on $\mathfrak{g}$ we can define a left invariant Finsler metric on G. It is easy to see that there do exist non-Riemannian ones if $\dim G \geqslant 2$. One can also consider the right action. However, we are interested in bi-invariant Finsler metrics. For this purpose, we consider the product group $G \times G$ and the subgroup

$$G^* = \{(g, g) \in G \times G | g \in G\}.$$

Then $G \times G/G^*$ is isomorphic to G under the mapping

$$(g_1, g_2)G^* \mapsto g_1 g_2^{-1}.$$

Under this isomorphism, a Finsler metric on G is bi-invariant if and only if the corresponding Finsler metric on $G \times G/G^*$ is $G \times G$-invariant. Now $G \times G/G^*$ can be viewed as a reductive homogeneous manifold with the decomposition

$$(x, y) = \left(\tfrac{1}{2}(x + y), \tfrac{1}{2}(x + y)\right) + \left(\tfrac{1}{2}(x - y), -\tfrac{1}{2}(x - y)\right), \qquad x, y \in \mathfrak{g}.$$

Let $\mathfrak{h} = \{(x, x) \in \mathfrak{g} + \mathfrak{g} | x \in \mathfrak{g}\}$, $\mathfrak{m} = \{(x, -x) | x \in \mathfrak{g}\}$. Then by theorem 1.3 we have

Proposition 2.1. *Let G be a connected Lie group. Then there is a one-to one correspondence between the bi-invariant Finsler metric on G and the Minkowski norm F on $\mathfrak{m}$ such that $\{\mathfrak{g} + \mathfrak{g}, \mathfrak{h}, F\}$ is a Minkowski Lie pair.*

Since $\mathfrak{h}$ is isomorphic to $\mathfrak{g}$ as a Lie algebra under the mapping $\sigma : (x, x) \mapsto x$ and $\mathfrak{m}$ is linear isomorphic to $\mathfrak{g}$ as a vector space under the mapping $\tau : (x, -x) \mapsto x$. We can restate proposition 2.1 without introducing $\mathfrak{h}$ and $\mathfrak{m}$. We first give a definition.

Definition 2.1. *Let $\mathfrak{g}$ be a real Lie algebra, F be a Minkowski norm on $\mathfrak{g}$. Then $\{\mathfrak{g}, F\}$ (or simply $\mathfrak{g}$) is called a* Minkowski Lie algebra *if the following condition is satisfied:*

$$g_y([x, u], v) + g_y(u, [x, v]) + 2C_y([x, y], u, v) = 0,$$

where $y \in \mathfrak{g} - \{0\}$, $x, u, v \in \mathfrak{g}$.

Proposition 2.2. *Let $\mathfrak{g}, \mathfrak{h}, \mathfrak{m}$ be as above. Then a Minkowski norm F on $\mathfrak{m}$ makes $\{\mathfrak{g} + \mathfrak{g}, \mathfrak{h}, F\}$ a Minkowski Lie pair if and only if the induced (by τ) Minkowski norm on $\mathfrak{g}$ makes $\{\mathfrak{g}, F\}$ a Minkowski Lie algebra.*

The proof is easy and we omit it.

In summarizing, we have proved the following:

Theorem 2.3. *Let G be a connected Lie group. Then there exists a bi-invariant Finsler metric on G if and only if there exists a Minkowski norm F on $\mathfrak{g}$ such that $\{\mathfrak{g}, F\}$ is a Minkowski Lie algebra.*

3. Geodesics and flag curvatures

In general, it is very difficult to describe explicitly the connections, geodesics and flag curvatures of a Finsler metric. However, for some special invariant Finsler metrics on a homogeneous manifold, we can obtain the explicit formula. We first recall a definition of Kobayashi and Nomizu.

Definition 2.1 (cf [6]). *Let G/H be a homogeneous manifold with an invariant (indefinite) Riemannian metric g. Then G/H is called* naturally reductive *if there exists an* $\mathrm{Ad}(H)$-*invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ such that*

$$B_1(x, [z, y]_{\mathfrak{m}}) + B_1([z, x]_{\mathfrak{m}}, y) = 0, \qquad x, y, z \in \mathfrak{m},$$

where B_1 is the bilinear form on $\mathfrak{m}$ induced by g and $[\,,\,]_{\mathfrak{m}}$ is the projection to $\mathfrak{m}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. A homogeneous manifold G/H with an invariant Finsler metric F is called naturally reductive *if there exists an invariant Riemannian metric g on G/H such that $(G/H, g)$ is naturally reductive and the connections of g and F coincide.*

Theorem 2.1. *Let G/H be a homogeneous manifold with an invariant Finsler metric F such that $(G/H, F)$ is naturally reductive with the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Then*

(1) The geodesics of G/H through o are $\exp(tx)\cdot o$, $x\in\mathfrak{m}$.

(2) The curvature tensor of F is given

$$(R(x,y)z)_o=\frac{1}{4}[x,[y,z]_{\mathfrak{m}}]_{\mathfrak{m}}-\frac{1}{4}[y,[x,z]_{\mathfrak{m}}]_{\mathfrak{m}}-\frac{1}{2}[[x,y]_{\mathfrak{m}},z]_{\mathfrak{m}}-[[x,y]_{\mathfrak{h}},z],$$

where $x,y,z\in\mathfrak{m}$.

(3) Let $y\in\mathfrak{m}$ and P be a plane in $\mathfrak{m}$ containing y. Then the flag curvature of the flag (P,y) is given by

$$K(P,y)=-\frac{1}{4}g_y([u,[v,u]_{\mathfrak{m}}]_{\mathfrak{m}},v)-\frac{1}{2}g_y([[v,u]_{\mathfrak{m}},u]_{\mathfrak{m}},v)-g_y([[v,u]_{\mathfrak{h}},u]_{\mathfrak{m}},v),$$

where $u=\frac{y}{\sqrt{g_y(y,y)}}$ and u,v is an orthonormal basis of P with respect to g_y.

Proof. Since G/H is naturally reductive, its connection is linear on M and has the same geodesics as the canonical connection (cf [6]). Therefore (1) follows. The formula of the connection of a naturally reductive homogeneous Riemannian manifold is given in [6]. Equation (3) is the direct consequence of (2) and the definition of flag curvature. □

As an explicit example, we consider the irreducible Riemannian globally symmetric spaces. Let $(G/H,g)$ be an irreducible Riemannian globally symmetric space with the rank $\geqslant 2$. Then Szabó [6] proved that there exists invariant non-Riemannian Finsler metric on G/H and each such metric is of Berwald type with the same connection as g. Since $(G/H,g)$ is naturally reductive, theorem 3.1 is applicable. Let $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ be the decomposition with respect to the canonical involution of G/H. Then we have

$$[\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m},\qquad [\mathfrak{m},\mathfrak{m}]\subset\mathfrak{h}.$$

Therefore the geodesics through o are $\exp tx\cdot o$, $x\in\mathfrak{m}$. The curvature tensor at o is given by

$$R_o(u,v)w=-[[u,v],w].$$

And the flag curvature of the flag (P,y) is given by

$$K(P,y)=-g_y([[u,v],u],v),$$

where the definition of u,v is similar to that of theorem 2.1.

4. Existence and examples

In the previous sections, we have investigated the general geometric properties of invariant Finsler metric on G/H. The problem is whether there exists a non-Riemannian invariant Finsler metric on a given homogeneous manifold. We will give some partial answers to this problem. Finally, some explicit examples are given.

Theorem 4.1. *Let G/H be a homogeneous manifold with H compact. Suppose the adjoint representation of H on $\mathfrak{g}/\mathfrak{h}$ is not irreducible. Then there exists an invariant non-Riemannian Finsler metric on G/H. In particular, if G is a connected compact Lie group which is not simple, then there exists a bi-invariant non-Riemannian Finsler metric on it.*

Proof. We use a similar method to Szabó (cf [7]). Since H is compact, the representation of H on $\mathfrak{g}/\mathfrak{h}$ admits an invariant inner product $\langle\,,\rangle$. Therefore $\mathfrak{g}/\mathfrak{h}$ has the decomposition

$$\mathfrak{g}/\mathfrak{h}=V_0+V_1+\cdots+V_n,$$

where V_0 is the subspace of fixed points of $\mathrm{Ad}(H)$ and V_i, $i=1,2,\ldots,n$ are irreducible invariant subspaces. With this assumption, we have the following cases:

(1) $n \geqslant 1$. In this case we construct a Minkowski norm F on $\mathfrak{g}/\mathfrak{h}$ as follows:

$$F(X) = \sqrt{|X|^2 + \sqrt[s]{|X_0|^{2s} + |X_1|^{2s} + \cdots + |X_n|^{2s}}},$$

where $|\cdot|$ denotes the length with respect to $\langle\, ,\rangle$, $X = X_0 + \cdots + X_n$ is the decomposition of X corresponding to the above decomposition of $\mathfrak{g}/\mathfrak{h}$ and s is an integer greater than 2. Then it is easy to check that F satisfies the condition of corollary 1.2 and hence defines an invariant Finsler metric on G/H which is non-Riemannian (cf [7]).

(2) $n = 0$. Then $\dim V_0 \geqslant 2$. In this case any invariant non-Euclidean Minkowski norm F_0 on V_0 (if it exists) is invariant under $\mathrm{Ad}(H)$. Therefore there exists an invariant non-Riemannian Finsler metric on G/H.

The last conclusion is the consequence of the first conclusion and theorem 2.1. □

The following result is due to Szabó ([7]).

Example 3.1. Let $(G/H, g)$ be an irreducible Riemannian symmetric space. Then we have

(1) if the rank of G/H is 1, then there does not exist an invariant non-Riemannian Finsler metric on it.

(2) If the rank of G/H is $\geqslant 2$, then there exists infinitely many different (that is, not isometric to each other) invariant non-Riemannian Finsler metrics on G/H and each such metric has the same connection as g.

By shifting the group H to a proper subgroup of it, we can find many more examples as follows.

Example 3.2. Let G/H be a globally symmetric Riemannian manifold of dimension $\geqslant 2$. Suppose

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$$

is the decomposition of $\mathfrak{g}$ with respect to the involution of G/H at the origin o. Let $\mathfrak{m}_1$ be any non-zero proper subspace of $\mathfrak{m}$. Let

$$H_1 = \{h \in H | \mathrm{Ad}(h)X = X,\ \forall X \in \mathfrak{m}_1\}.$$

Then by theorem 4.1, there exist invariant non-Riemannian Finsler metrics on G/H_1.

As an explicit example, let us consider $S^n = SO(n+1)/SO(n), n \geqslant 2$. We have $\mathfrak{g} = \mathfrak{so}(n+1), \mathfrak{h} = \mathfrak{so}(n)$,

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & \alpha \\ -\alpha^t & 0 \end{pmatrix} \middle| \alpha \in \mathbb{R}^n \right\}. \tag{3.1}$$

For an integer q, $1 \leqslant q \leqslant n-1$, let $\mathfrak{m}_q$ be the subspace of $\mathfrak{m}$ with

$$\alpha = \begin{pmatrix} \alpha_{n-q} \\ 0 \end{pmatrix}, \qquad \alpha_{n-q} \in \mathbb{R}^{n-q}$$

in (3.1). The subgroup of $SO(n)$ which leaves each point of $\mathfrak{m}_q$ fixed is $SO(q) \hookrightarrow SO(n)$ as

$$SO(q) = \left\{ \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix} \middle| A \in SO(q) \right\}.$$

Then by the above argument, there exist invariant non-Riemannian Finsler metrics on $M = SO(n+1)/SO(q)$. In fact, using the method of theorem 4.1, we can write down an explicit invariant Finsler metric on M. The tangent space $T_o(M)$ can be identified with

$$V = \left\{ \left. \begin{pmatrix} \beta & 0 & \alpha \\ 0 & 0 & \gamma \\ -\alpha^t & -\gamma^t & 0 \end{pmatrix} \right| \beta \in \mathfrak{so}(n-q), \alpha \in \mathbb{R}^{n-q}, \gamma \in \mathbb{R}^q \right\}.$$

The fixed point set of $SO(q)$ is

$$V_0 = \left\{ \left. \begin{pmatrix} \beta & 0 & \alpha \\ 0 & 0 & 0 \\ -\alpha^t & 0 & 0 \end{pmatrix} \right| \beta \in \mathfrak{so}(n-q), \alpha \in \mathbb{R}^{n-q} \right\}.$$

And another irreducible invariant subspace of $SO(q)$ is

$$V_1 = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \gamma \\ 0 & -\gamma^t & 0 \end{pmatrix} \right| \gamma \in \mathbb{R}^q \right\}.$$

In V, we take the $SO(q)$-invariant inner product defined by

$$\langle A_1, A_2 \rangle = \mathrm{Tr}\big(A_1^t A_2\big), \qquad A_1, A_2 \in V.$$

Then for any integer $s \geqslant 2$, we can define an invariant non-Riemannian Finsler metric on M by

$$F_o(A) = \sqrt{\mathrm{Tr}(\beta^t\beta) + 2\alpha^t\alpha + 2\gamma^t\gamma + \sqrt[s]{(\mathrm{Tr}(\beta^t\beta) + 2\alpha^t\alpha)^s + (2\gamma^t\gamma)^s}},$$

where

$$A = \begin{pmatrix} \beta & 0 & \alpha \\ 0 & 0 & \gamma \\ -\alpha^t & -\gamma^t & 0 \end{pmatrix} \in V.$$

Note that

$$SO(n+1)/SO(q) = (SO(n+1)/SO(q) \times SO(p)) \times SO(p),$$

where $p = n+1-q$. Therefore M is a fibre bundle over the Grassmannian manifold $G_{p,q}(\mathbb{R})$ with fibres $SO(p)$.

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